Figure 3 shows the instantaneous images obtained from case E1A. Large-scale eddies formed at the edge of the jet near the injector wall influence the behavior of the bow shock below the point of intersection with the separation shock in case E1A, though not to the degree that was observed in Fig. 2. The images in Fig. 3 give some evidence of the curvature changes induced by the eddies. Also, the third image shows the shock lifting phenomenon that appeared in case C1A. The bow shock standoff distance is approximately $0.25d_{\mathrm{eff}}$ (50% closer to the injector leading edge than for case C1A). This smaller standoff distance is explained by the asymmetric vortex development associated with the elliptical injector geometry leading to more rapid spreading in the minor-axis direction than in the major-axis direction.[10] Apparently, the jet fluid expands more in the spanwise direction (z) than in the streamwise direction (x) at the exit plane of the elliptical nozzle, allowing the shock to stand closer to the jet exit and still accomplish the necessary pressure correction. In the region above the point of intersection of the bow and separation shocks, small fluctuations in the bow shock's position occur. However, the images shown in Fig. 3 indicate a weaker bow shock in case E1A than in case C1A as inferred by its slope. The ensemble of images obtained for case E1A indicates that the bow shock intersects the upper edge of the image at streamwise positions between $-0.25 \leq x/d_{\mathrm{eff}} \leq 0$. Thus, in addition to being weaker than the wave in case C1A, the bow shock in case E1A fluctuates over a smaller spatial range.

The upstream extent of the separation shock in case E1A falls between $-2.75 \leq x/d_{\mathrm{eff}} \leq -1.75$, whereas the point of intersection of the bow and separation shocks is between 1.1 and 2.1 effective diameters above the bottom wall. These values are significantly lower than those obtained from case C1A; thus, the area beneath the separation shock associated with the elliptical injector is smaller than in circular injection at the spanwise centerline of the jet/crossflow interaction. Therefore, injection through the elliptical nozzle potentially reduces any hot-spot phenomenon associated with injection through a circular nozzle oriented perpendicular to the supersonic freestream.

Conclusions

In flowfields created by transverse injection into supersonic crossflows, the bow and separation shocks formed upstream of the injectant plume are dominant features. In the present investigation, the interaction between these features and the large-scale eddies that develop at the jet/freestream interface has been examined. Results indicate that the large structures strongly influence the near-wall behavior of the bow shock, often resulting in severe curvature changes and positional fluctuations. The eddies exert a weaker influence on the bow shock further away from the wall. Lifting of the bow shock has been observed when the approaching boundary layer is relatively thick. In these instances, injectant and freestream fluid mix subsonically upstream of the injector orifice, thereby exacerbating the hot-spot phenomenon found in reacting transverse injection flowfields. Finally, the injector geometry strongly affected the upstream separation zone, the bow shock standoff distance, and the strength of the bow shock. Elliptical injection with the major axis aligned with the freestream flow resulted in a smaller separation zone and standoff distance and a weaker bow shock compared with circular injection.

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Proving Algorithm Symmetry for Flows Exhibiting Symmetry Breaking

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Introduction

THIS Note is prompted by the need to demonstrate the symmetry of the conical Navier–Stokes algorithm used by Dusing and Orkwis[1] in studying the formation of vortex asymmetry in flows over cones at incidence. Algorithm-related asymmetry because of approximate factorization has been demonstrated previously for the Beam–Warming algorithm in its diagonal form by Levy et al.[2] The symmetry of the current algorithm was studied to avoid similar spurious computations.

In general, this issue is important whenever flows that involve symmetry breaking are under consideration, namely, flow over a cone, flow over a circular cylinder, flow in a divergent nozzle of rectangular cross section, flow along a corner, and so forth.[3–6] These flows undergo bifurcation at some critical value of a parameter (or combination of parameters), e.g., Reynolds number in the cylinder case, divergence angle in the nozzle case. Numerical simulations can provide disturbance-free base flows for studying the stability properties of the symmetric flow only if the algorithm employed in the simulation is free from asymmetric bias. In other words, we expect to capture the symmetric solution using a bias-free algorithm, given symmetric initial and boundary conditions. However, this would be possible only in the absence of round-off errors. For the purposes of reliable linear stability analyses of symmetric base flows, solutions converged close to machine accuracy are still acceptable, but it is necessary to employ a symmetric algorithm to achieve this. It is therefore imperative that algorithm symmetry be ensured before such calculations are attempted. This can be done analytically, and a simple illustration is presented.

In this Note, the case of flow over a cone at an angle of attack is considered. The thin-layer Navier–Stokes (TLNS) equations are to be solved on a conical grid using the two-step approximate factorization scheme of Anderson[7] as modified by Vanden and Belk.[8] Further details of the algorithm can be found in Ref. 1. The goal of this Note is to check the symmetry of the algorithm by analytical means. To this effect, the symbolic manipulation software MACSYMA is used in part.

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Governing Equations and Boundary Conditions

The three-dimensional TLNS equations can be written in conservation form as

$$\frac{\partial Q}{\partial t} + \frac{\partial(F_i)}{\partial \xi} + \frac{\partial(G_i - S_v)}{\partial \eta} + \frac{\partial(H_i)}{\partial \zeta} = 0 \tag{1}$$

where $Q = [\rho \;\; \rho u \;\; \rho v \;\; \rho w \;\; e]^T$ and F_i, G_i, and H_i are the inviscid flux vectors given by

$$F_i = \begin{bmatrix} \rho U \\ \rho u U + \xi_x p \\ \rho v U + \xi_y p \\ \rho w U + \xi_z p \\ (e+p)U \end{bmatrix}, \qquad G_i = \begin{bmatrix} \rho V \\ \rho u V + \eta_x p \\ \rho v V + \eta_y p \\ \rho w V + \eta_z p \\ (e+p)V \end{bmatrix}$$

$$H_i = \begin{bmatrix} \rho W \\ \rho u W + \zeta_x p \\ \rho v W + \zeta_y p \\ \rho w W + \zeta_z p \\ (e+p)W \end{bmatrix} \tag{2}$$

S_v is the only viscous flux vector that is retained (details in Ref. 9), ξ is the radial direction, η is the direction normal to the body, and ζ is the azimuthal direction. A steady form of the above equations is solved in a single crossflow (η, ζ) plane of a conical grid in which the ξ-direction lines are rays from the cone tip. Constant property boundary conditions are enforced along these rays (the conical flow assumption) to convert the three-dimensional solver to a conical form. Physical boundary conditions are required only on the surface and far-field boundaries.

Numerical Method

The two-pass factorization for the generalized three-dimensional TLNS equations can be derived from Eq. (1) starting with a simple forward difference for the temporal term, rewriting the equation in delta form, linearizing the $(n+1)$-level flux Jacobians, and neglecting the viscous terms on the left-hand side:

$$\left[I + \Delta\tau \left(\frac{\partial A}{\partial \xi}\right)^n + \Delta\tau \left(\frac{\partial B}{\partial \eta}\right)^n + \Delta\tau \left(\frac{\partial C}{\partial \zeta}\right)^n \right] \Delta^n Q$$

$$= -\Delta\tau \left(\frac{\partial F_i^n}{\partial \xi} + \frac{\partial (G_i^n - S_v^n)}{\partial \eta} + \frac{\partial H_i^n}{\partial \zeta} \right) \tag{3}$$

where F_i, G_i, H_i, and S_v are as in Eq. (1), $\Delta^n Q$ is the vector of changes in the conserved variables, $\Delta\tau$ is the computational time step, and A, B, and C are the inviscid-flux Jacobians, $(\partial F_i/\partial Q)$, $(\partial G_i/\partial Q)$, and $(\partial H_i/\partial Q)$, respectively, which when differenced are understood to act as operators on $\Delta^n Q$. The scheme is now semi-implicit because the second approximation causes the viscous terms to be lagged. However, this does not affect the solution because convergence to a steady state is defined as a zero right-hand side (RHS) to machine accuracy. Consequently, the n-level-flux Jacobians can be split, which allows the linearized discrete governing equations to be written as the approximate two-pass factorization developed by Anderson[7] and later used by Vanden and Belk[8]:

$$\left(I + \Delta\tau \nabla_\xi A^+ + \Delta\tau \nabla_\eta B^+ + \Delta\tau \Delta_\eta B^-\right)\left(I + \Delta\tau \Delta_\xi A^-\right.$$

$$\left. + \Delta\tau \nabla_\zeta C^+ + \Delta\tau \Delta_\zeta C^-\right)\Delta^n Q = (\text{RHS})^n \tag{4}$$

where $A^\pm$, $B^\pm$, and $C^\pm$ are the Steger–Warming flux-vector-split Jacobians; $(\text{RHS})^n$ is the discretized RHS of Eq. (3); $\Delta\xi$, $\Delta\eta$, and $\Delta\zeta$ are assumed to be unity; and Δ_ξ and ∇_ξ, etc., are the standard two-point-forward and -backward difference operators $\Delta_\xi() = ()_{i+1,j,k} - ()_{i,j,k}$ and $\nabla_\xi() = ()_{i,j,k} - ()_{i-1,j,k}$. The bracketed terms are considered operators acting on $\Delta^n Q$ (i.e., $\Delta_\xi A^- \Delta^n Q = [A^- \Delta^n Q]_{i+1,j,k} - [A^- \Delta^n Q]_{i,j,k}$). For simplicity, we assume that the RHS is discretized using central differencing. In practice, however, more complicated discretizations are employed, and such a case is analyzed in Ref. 9.

The factorization in Eq. (4) requires the solution of two block tetradiagonal systems of equations,

$$\left(I + \Delta\tau \nabla_\xi A^+ + \Delta\tau \nabla_\eta B^+ + \Delta\tau \Delta_\eta B^-\right) Q^* = (\text{RHS})^n \tag{5}$$

$$\left(I + \Delta\tau \Delta_\xi A^- + \Delta\tau \nabla_\zeta C^+ + \Delta\tau \Delta_\zeta C^-\right) \Delta^n Q = Q^* \tag{6}$$

which can be solved by employing a block tridiagonal solver with forward ξ-direction sweeps for step 1, and backward ξ-direction sweeps for step 2. However, the conical solver uses the above equations in an abbreviated form, i.e., only one plane of cells is considered. Therefore, $\Delta_\xi A^- \approx -A^-_{i,j,k}$ and $\nabla_\xi A^- \approx A^-_{i,j,k}$. Note that the above factorization is expected to be symmetric in ζ because both C^+ and C^- are in the same factor.

Symmetry Proof

As stated in "Governing Equations and Boundary Conditions," the conical solver was developed from a generalized coordinate three-dimensional solver by simplifying the algorithm when applied on a conical grid. If Cartesian coordinates are used to describe the physical space, the x direction corresponds to the cone axis; the y direction is vertical, such that incidence is achieved by a pitch-up in the (x, y) plane; and the z direction is orthogonal to the (x, y) plane. For this geometry and grid, symmetry in the crossplane requires that the mirror-image grid points (i, j, k) and $(\hat{i}, \hat{j}, \hat{k})$ at locations (x, y, z) and $(x, y, -z)$, respectively, must have

$$Q_{i,j,k} = M Q_{\hat{i},\hat{j},\hat{k}} \tag{7}$$

where

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ e \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Clearly, $M = M^{-1}$. Also, the metrics for a conical grid are related in the following manner:

$$(\xi_x)_{i,j,k} = (\xi_x)_{\hat{i},\hat{j},\hat{k}}$$

$$(\xi_y)_{i,j,k} = (\xi_y)_{\hat{i},\hat{j},\hat{k}} = (\xi_z)_{i,j,k} = (\xi_z)_{\hat{i},\hat{j},\hat{k}} = 0$$

$$(\eta_x)_{i,j,k} = (\eta_x)_{\hat{i},\hat{j},\hat{k}}, \qquad (\eta_y)_{i,j,k} = (\eta_y)_{\hat{i},\hat{j},\hat{k}} \tag{8}$$

$$(\eta_z)_{i,j,k} = -(\eta_z)_{\hat{i},\hat{j},\hat{k}}, \qquad (\zeta_x)_{i,j,k} = -(\zeta_x)_{\hat{i},\hat{j},\hat{k}}$$

$$(\zeta_y)_{i,j,k} = -(\zeta_y)_{\hat{i},\hat{j},\hat{k}}, \qquad (\zeta_z)_{i,j,k} = (\zeta_z)_{\hat{i},\hat{j},\hat{k}}$$

Consequently, the cell contravariant velocities are related as

$$U_{i,j,k} = U_{\hat{i},\hat{j},\hat{k}}, \qquad V_{i,j,k} = V_{\hat{i},\hat{j},\hat{k}}, \qquad W_{i,j,k} = -W_{\hat{i},\hat{j},\hat{k}} \tag{9}$$

To prove the symmetry of the algorithm, we start with the assumption that the solution from the previous iteration $(n-1)$ is symmetric, i.e., it satisfies relation (7) (for $n = 1$, this implies symmetric initial condition), and prove that

$$(\text{RHS})^n_{i,j,k} = M (\text{RHS})^n_{\hat{i},\hat{j},\hat{k}} \tag{10}$$

Consequently, we expect the intermediate solution obtained by solving Eq. (5) to be symmetric

$$Q^*_{i,j,k} = M Q^*_{\hat{i},\hat{j},\hat{k}} \tag{11}$$

and since $Q^{n+1} = Q^n + \Delta^n Q$, the solution at the $(n+1)$ level will be symmetric, provided the change obtained by solving Eq. (6) is also symmetric:

$$\Delta^n Q_{i,j,k} = M \Delta^n Q_{\hat{i},\hat{j},\hat{k}} \tag{12}$$

We start the proof with the interesting observation that although point (i, j, k) corresponds to point $(\hat{i}, \hat{j}, \hat{k})$, the azimuthal ordering prescribes the correspondence $(i, j, k+1)$ to $(\hat{i}, \hat{j}, \hat{k}-1)$ and $(i, j, k-1)$ to $(\hat{i}, \hat{j}, \hat{k}+1)$. Following relations (8) and (9), the fluxes in Eq. (2) obey $X_{i,j,k} = MX_{\hat{i},\hat{j},\hat{k}}$ (where $X = F$ or G), $H_{i,j,k} = -MH_{\hat{i},\hat{j},\hat{k}}$, and $H_{i,j,k\pm1} = -MH_{\hat{i},\hat{j},\hat{k}\mp1}$. Hence, it is easy to see that a simple central difference makes $(\mathrm{RHS})^n$ in Eq. (5) symmetric because

$$\left(\frac{\partial X}{\partial \zeta}\right)_{i,j,k} = M\left(\frac{\partial X}{\partial \zeta}\right)_{\hat{i},\hat{j},\hat{k}}$$

where $X = F, G,$ or H. This establishes relation (10). The viscous term S_v has not been considered here for brevity, but it is shown in Ref. 9. It is difficult to prove this for a higher-order flux-limited Riemann solver type discretization, although this has been shown in Ref. 9 using MACSYMA.

The left-hand-side expressions of Eq. (3) also involve the parity reversal in k-ordering. However, symmetry is easily demonstrated using the following relationships (proved in Ref. 9):

$$A^{\pm}_{i,j,k}M = MA^{\pm}_{\hat{i},\hat{j},\hat{k}}, \qquad B^{\pm}_{i,j,k}M = MB^{\pm}_{\hat{i},\hat{j},\hat{k}}$$
$$C^{\pm}_{i,j,k\pm\delta}M = -MC^{\mp}_{\hat{i},\hat{j},\hat{k}\mp\delta} \tag{13}$$

where $\delta = -1, 0, 1$. The third relation reflects the reversal of ordering in k and $\hat{k}$. It is clear that symmetry of the algorithm cannot be guaranteed unless C^+ and C^- are included in the same factor.

Consider next the discrete form of Eq. (5) at the point (i, j, k):

$$Q^*_{i,j,k} + \Delta\tau\big([A^+Q^*]_{i,j,k} - [A^+Q^*]_{i-1,j,k} + [B^+Q^*]_{i,j,k}$$
$$- [B^+Q^*]_{i,j-1,k} + [B^-Q^*]_{i,j+1,k} - [B^-Q^*]_{i,j,k}\big)$$
$$= (\mathrm{RHS})^n_{i,j,k} \tag{14}$$

Premultiply the discrete form of Eq. (5) at the point $(\hat{i}, \hat{j}, \hat{k})$, and assume relation (10) to obtain:

$$MQ^*_{\hat{i},\hat{j},\hat{k}} + \Delta\tau\big([MA^+Q^*]_{\hat{i},\hat{j},\hat{k}} - [MA^+Q^*]_{\hat{i}-1,\hat{j},\hat{k}}$$
$$+ [MB^+Q^*]_{\hat{i},\hat{j},\hat{k}} - [MB^+Q^*]_{\hat{i},\hat{j}-1,\hat{k}} + [MB^-Q^*]_{\hat{i},\hat{j}+1,\hat{k}}$$
$$- [MB^-Q^*]_{\hat{i},\hat{j},\hat{k}}\big) = M\,(\mathrm{RHS})^n_{\hat{i},\hat{j},\hat{k}} = (\mathrm{RHS})^n_{i,j,k} \tag{15}$$

Using the relations (13) in Eq. (15) gives

$$MQ^*_{\hat{i},\hat{j},\hat{k}} + \Delta\tau\big(A^+_{i,j,k}MQ^*_{\hat{i},\hat{j},\hat{k}} - A^+_{i-1,j,k}MQ^*_{\hat{i}-1,\hat{j},\hat{k}}$$
$$+ B^+_{i,j,k}MQ^*_{\hat{i},\hat{j},\hat{k}} - B^+_{i,j-1,k}MQ^*_{\hat{i},\hat{j}-1,\hat{k}} + B^-_{i,j+1,k}MQ^*_{\hat{i},\hat{j}+1,\hat{k}}$$
$$- B^-_{i,j,k}MQ^*_{\hat{i},\hat{j},\hat{k}}\big) = (\mathrm{RHS})^n_{i,j,k} \tag{16}$$

Subtracting Eq. (16) from Eq. (14) yields

$$\big[Q^*_{i,j,k} - MQ^*_{\hat{i},\hat{j},\hat{k}}\big] + \Delta\tau\big(A^+_{i,j,k}\big[Q^*_{i,j,k} - MQ^*_{\hat{i},\hat{j},\hat{k}}\big]$$
$$- A^+_{i-1,j,k}\big[Q^*_{i-1,j,k} - MQ^*_{\hat{i}-1,\hat{j},\hat{k}}\big] + B^+_{i,j,k}\big[Q^*_{i,j,k} - MQ^*_{\hat{i},\hat{j},\hat{k}}\big]$$
$$- B^+_{i,j-1,k}\big[Q^*_{i,j-1,k} - MQ^*_{\hat{i},\hat{j}-1,\hat{k}}\big] + B^-_{i,j+1,k}\big[Q^*_{i,j+1,k}$$
$$- MQ^*_{\hat{i},\hat{j}+1,\hat{k}}\big] - B^-_{i,j,k}\big[Q^*_{i,j,k} - MQ^*_{\hat{i},\hat{j},\hat{k}}\big]\big) = 0 \tag{17}$$

for all (i, j, k). If the flux Jacobians are nonsingular, the terms within brackets must be identically zero at all points for Eq. (17) to hold. Then the first, second, fourth, and seventh terms within brackets imply relation (11), which proves the symmetry of the intermediate solution.

To prove the last step, we consider the discretized form of Eq. (6) at the point (i, j, k):

$$\Delta^n Q_{i,j,k} + \Delta\tau\big([A^-\Delta^n Q]_{i+1,j,k} - [A^-\Delta^n Q]_{i,j,k}$$
$$+ [C^+\Delta^n Q]_{i,j,k} - [C^+\Delta^n Q]_{i,j,k-1} + [C^-\Delta^n Q]_{i,j,k+1}$$
$$- [C^-\Delta^n Q]_{i,j,k}\big) = Q^*_{i,j,k} \tag{18}$$

Premultiply the discrete form of Eq. (6) at the point $(\hat{i}, \hat{j}, \hat{k})$, and assume relation (11) to obtain

$$M\Delta^n Q_{\hat{i},\hat{j},\hat{k}} + \Delta\tau\big([MA^-\Delta^n Q]_{\hat{i}+1,\hat{j},\hat{k}} - [MA^-\Delta^n Q]_{\hat{i},\hat{j},\hat{k}}$$
$$+ [MC^+\Delta^n Q]_{\hat{i},\hat{j},\hat{k}} - [MC^+\Delta^n Q]_{\hat{i},\hat{j},\hat{k}-1} + [MC^-\Delta^n Q]_{\hat{i},\hat{j},\hat{k}+1}$$
$$- [MC^-\Delta^n Q]_{\hat{i},\hat{j},\hat{k}}\big) = MQ^*_{\hat{i},\hat{j},\hat{k}} = Q^*_{i,j,k} \tag{19}$$

Using the relations (13) in Eq. (19) gives

$$M\Delta^n Q_{\hat{i},\hat{j},\hat{k}} + \Delta\tau\big(A^-_{i+1,j,k}M\Delta^n Q_{\hat{i}+1,\hat{j},\hat{k}} - A^-_{i,j,k}M\Delta^n Q_{\hat{i},\hat{j},\hat{k}}$$
$$- C^-_{i,j,k}M\Delta^n Q_{\hat{i},\hat{j},\hat{k}} + C^-_{i,j,k+1}M\Delta^n Q_{\hat{i},\hat{j},\hat{k}-1}$$
$$- C^+_{i,j,k-1}M\Delta^n Q_{\hat{i},\hat{j},\hat{k}+1} + C^+_{i,j,k}M\Delta^n Q_{\hat{i},\hat{j},\hat{k}}\big) = Q^*_{i,j,k} \tag{20}$$

Subtracting Eq. (20) from Eq. (18) yields

$$\big[\Delta^n Q_{i,j,k} - M\Delta^n Q_{\hat{i},\hat{j},\hat{k}}\big] + \Delta\tau\big(A^-_{i+1,j,k}\big[\Delta^n Q_{i+1,j,k}$$
$$- M\Delta^n Q_{\hat{i}+1,\hat{j},\hat{k}}\big] - A^-_{i,j,k}\big[\Delta^n Q_{i,j,k} - M\Delta^n Q_{\hat{i},\hat{j},\hat{k}}\big]$$
$$- C^-_{i,j,k}\big[\Delta^n Q_{i,j,k} - M\Delta^n Q_{\hat{i},\hat{j},\hat{k}}\big] + C^-_{i,j,k+1}\big[\Delta^n Q_{i,j,k+1}$$
$$- M\Delta^n Q_{\hat{i},\hat{j},\hat{k}-1}\big] - C^+_{i,j,k-1}\big[\Delta^n Q_{i,j,k-1} - M\Delta^n Q_{\hat{i},\hat{j},\hat{k}+1}\big]$$
$$+ C^+_{i,j,k}\big[\Delta^n Q_{i,j,k} - M\Delta^n Q_{\hat{i},\hat{j},\hat{k}}\big]\big) = 0$$

By similar arguments as before, the first, third, fourth, and seventh terms within brackets imply relation (12), which consequently proves the symmetry of the overall algorithm.

Summary

Algorithm symmetry is an important issue that merits consideration when computing flows that exhibit symmetry-breaking bifurcation. A simple guideline is provided to check symmetry of algorithms, in particular, approximate factorization schemes. The TLNS algorithm of Anderson,[7] modified by Vanden and Belk,[8] applied to the specific case of flow over a cone at incidence is symmetric. A simplified version of the analytical proof (using symbolic manipulation software) is outlined. It is suggested that such an effort should accompany any numerical simulation of flows exhibiting symmetry breaking.

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